Superthermal Electrons and Bernstein Waves in Jupiter's Inner Magnetosphere

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A theoretical model for the generation of banded electrostatic emissions by low density, superthermal electrons is developed for application to Jupiter's magnetosphere. The model employs a power law form for the energy dependence and a loss cone pitch angle distribution of the superthermals to drive convective instability of Bernstein modes. We concentrate on instability in the upper hybrid band and on lower harmonic bands below the upper hybrid frequency. A direct correspondence between spectral features of the 3/2's band and resonant superthermal electrons is found. The concept of a critical flux of resonant electrons able to provide 10 e-foldings of electric field amplification yields an explicit relation \( j_{x*} \sim p \) in terms of the background thermal electron pressure. This result is used to construct a theoretical/empirical model of thermal electron density and temperature from 6-20 \( R_J \) in the Jovian magnetosphere which suggests that the ion and electron temperatures satisfy \( T_e < T_i \sim 10 T_e \) in this region. Finally, wave ray paths are computed for propagation in the magnetic equator and in the meridional plane of a dipole magnetic field. These ray paths suggest that intense wave activity is tightly confined to a small latitudinal extent, \( \Lambda \approx \pm 4^\circ \), about the magnetic equator.

1. INTRODUCTION

Recent observations made by the Voyager spacecraft of electrostatic waves in Jupiter's magnetosphere [Scarf et al., 1979; Warwick et al., 1979; Gurnett et al., 1979c; Kurth et al., 1980] have afforded the opportunity for a penetrating look into plasma-physical processes occurring there. These spacecraft have already confirmed that whistler wave theory developed for the terrestrial magnetosphere [Kennel and Petschek, 1966] can be used as both a predictive and diagnostic tool for quantitative magnetospheric studies and have firmly established plasma wave measurement and attendant theory as an integral part of extraterrestrial magnetospheric investigation. Now observations of plasma waves above the electron gyrofrequency have revealed another class of emissions with direct terrestrial analogs and have extended the range of resonant wave-particle interaction energies which these wave measurements indirectly monitor.

At the Earth, high-altitude satellites have documented many of the basic characteristics of electrostatic waves associated with the electron gyrofrequency and plasma frequency [Kennel et al., 1970; Fredricks and Scarf, 1973; Moizer et al., 1973; Shaw and Gurnett, 1975]. In particular, recent ISEE and Geos measurements have emphasized the multiharmonic character of these waves generally referred to as odd half-harmonic emissions at \((n + 1/2)f_\omega\) [Christiansen et al., 1978; Gurnett et al., 1979a]. Only recently has a convincing case been given that in the magnetosphere these electron cyclotron harmonic (ECH) emissions are, indeed, of a general class known as Bernstein modes [Bernstein, 1958; Christiansen et al., 1978; Rönnmark et al., 1978].

Detailed studies of these waves at the Earth have revealed the following principal characteristics which we shall concentrate on: (1) intense emission (\( \geq 10 \) mV/m) at or near the upper hybrid frequency [Christiansen et al., 1978; Kurth, 1979; Kurth et al., 1979b], (2) moderately intense emission (1-10 mV/m) in the lower \( n = 1, 2, \) etc., gyroharmonic bands [Christiansen et al., 1978], and (3) a tight association of these waves with the magnetic equator [Christiansen et al., 1978; Gough et al., 1979]. To be sure, many examples can be found that are exceptions to the above outline. For example, intense upper hybrid emissions are found at mid-latitudes and also weak diffuse banded emissions may be found at all latitudes much of the time. However, on a statistical basis which gives more weight to intense emissions, the above characteristics are deemed to be fairly representative of ECH wave phenomena at Earth. At Jupiter, the above features are also salient although wave intensities measured by the Voyager flybys were somewhat smaller [Kurth et al., 1980].

The purpose of this paper is to formulate a theory which can address as much of the Jovian ECH phenomenology as possible. For the case of terrestrial emissions, theoretical development has progressed significantly and the basis for this present work is found in the papers of Young et al. [1973], Ashour-Abdalla and Kennel [1978], and Hubbard and Birmingham [1978]. For the case of the Jovian emissions, papers most relevant to our present endeavor are Hubbard and Birmingham [1978], Rönnmark et al. [1978], and Kurth et al. [1979a] which consider multiharmonic wave emissions driven by a low density superthermal electron population. Novel features that our paper introduces are the use of a power law form for the superthermal energy distribution and calculations of ray paths for the lower ECH bands. The power law form offers a flexibility to theoretical modelling that broadens the range of resonant wave-particle interaction energies possible at nonnegligible flux levels (a modified power law was also employed by Hubbard and Birmingham [1978]); it is an attractive model not simply because it is compatible with particle observations but, more cogently, its theoretical consequences are more consonant with wave spectral observations. As far as we know no ray path calculations near the equator have appeared at all in the literature. Their significance in this context of convective amplification has been demonstrated by Barbosa [1976] and by Maggs [1978] for electrostatic wave propagation in the topside ionosphere.

For application to Jupiter we employ the theory for diagnostic investigation of thermal and superthermal electron properties in the inner-middle magnetosphere. An explicit relation is found for the critical flux of resonant electrons (that...
which produces 10 $e$-foldings of wave amplitude) in terms of background thermal electron pressure. This procedure is similar to that employed by Barbosa and Coroniti [1976] for whistler waves in the inner magnetosphere; however, electrostatic waves carry that extra information concerning electron temperature and thus provide a unique probe of magnetospheric plasmas. Based on a comparison with measured energetic electron fluxes, we then construct a model of thermal electron temperature in the inner magnetosphere from 6–20 $R_e$ that reconciles the observations of the plasma waves with in situ plasma and spectroscopic measurements of electron temperature.

In section 2 we briefly review ECH wave observations at Jupiter and motivate the model for the electron distribution used in our calculations. Section 3 contains the results of our linear instability analysis for three gyroharmonic frequency bands. In section 4, ray path calculations are performed for the lower ECH bands. In section 5, we apply these results to Jupiter's inner-middle magnetosphere to obtain models for electron and ion particle properties that are both compatible with our instability analysis and consistent with published in situ Voyager measurements at discrete points in the magnetosphere. The reader may find it useful to peruse this section prior to the instability analysis in order to get an idea of the range of parameters that we shall be dealing with and an image of Jupiter's magnetosphere as we perceive it to be. Finally, section 6 summarizes our results and indicates the direction of our current research.

2. MODEL FOR SUPERThermal ELECTRONS

a. Review of Jupiter Wave Observations

Kurth et al. [1980] have described the basic features of ECH waves in Jupiter's magnetosphere. We note that Voyager 1 observed electrostatic waves on every crossing of the magnetic equator for distances $R \leq 23 R_e$. We shall concentrate on those observations relevant for the inner magnetosphere where the magnetic field may be taken as approximately dipolar. Each crossing was accentuated by banded emissions, the lowest $n = 1$ band being moderately intense, successive higher bands decreasing in intensity, and the upper hybrid band very conspicuous. Banded emissions above the upper hybrid frequency band were not detected quite possibly due to instrument limitations. The multiharmonic aspect of these waves is evident. That is, depending on some feature of the free-energy source, simultaneous excitation in all or some gyroharmonic bands occurs with a hierarchy in intensity of the upper hybrid, the lowest $n = 1$ band, the $n = 2$ band, etc., in that order.

Electron density measurements made by the Planetary Radio Astronomy group [Warwick et al., 1979] have indicated that large background electron densities exist in the vicinity of the Io plasma torus with a plasma thermal temperature $\sim 50$ eV [Bridge et al., 1979]. It seems reasonable that any model for wave generation be based, therefore, on a two-component electron distribution with the free energy for waves located in a low density, superthermal population. If the terms hot and cold (cool) label these two populations, this model then assumes $n_h/n_c \ll 1$. At the Earth, Hubbard and Birmingham [1978] have investigated this regime in order to explain excitation in the upper hybrid band, but invoke the inverse ratio to explain low harmonic waves. Rönnmark et al. [1978] argued that the low density superthermal model can explain excitation in all bands and that $n_h/n_c \ll 1$ is probably realized in all circumstances. This regime is also treated in the Curtis and Wu [1979] model since they invoke relativistic electrons for wave generation.

The Jupiter measurements have singled out the magnetic equator as an important location for wave growth. The largest latitudinal extent for ECH waves observed by Voyager 1 was $\pm 1.9^\circ$. This association with the magnetic equator suggests that a pancaked pitch angle distribution for the hot electrons is a likely possibility for the instability mechanism. It is important to note that the equatorial clustering of wave activity does not arise solely from the fact that particle fluxes maximize at the equator; an unrealistically high value for the anisotropy is required to confine particles to just $\pm 2^\circ$. Rather, the tight confinement to the equator strongly suggests that wave propagation and convective effects play a crucial role in the amplification process depending sensitively on the value of the wavevector component $k_z = k \cdot B_0/B_0$ [Barbosa, 1980].

b. Particle Model

In view of the above discussion, we assume an isotropic Maxwellian distribution for the cold background with density $n_c$ and temperature $T_c$. For the hot electrons we employ a model previously investigated by Barbosa and Coroniti [1976] for application to Jupiter,

$$f_h(v) = A \frac{\sin^2 \theta}{T_h^{n_h}}$$

where $\cos \theta = v \cdot B_0/v_B$, $B_0 = B_0 z$, $T = m v^2/2$, and $A$ is a normalization factor. If $j_{\perp}(T_e)$ is the unidirectional flux of electrons perpendicular to $B_0$ in units of $(c m^2 s sr)$, then a convenient normalization is

$$f_h(v) = \frac{1}{2} m^2 T_{h}^{n_h} j_{\perp}(T_e) \frac{\sin^2 \theta}{T_h^{n_h+1}}$$

and $j_{\perp}(T) = j_{\perp}(T_e)(T_e/T)^N$ where $N$ is the spectral index. The omnidirectional integral flux is then given as

$$J(>T) = \left[ \frac{2 \pi e^2}{\Gamma(M+1)} \frac{\Gamma(M+3/2)}{\Gamma(M+1)} \right] (N-1)^{-1} T_{\perp}(T)$$

and the integral number density is

$$N(>T) = \left[ \frac{2 \pi e^2}{\Gamma(M+1)} \frac{\Gamma(M+3/2)}{\Gamma(M+1)} \right] (N-1) T_{\perp}(T)^{-1}$$

If $N = 1$ ($N = 1/2$), the term in $T_{\perp}(T)$ should be substituted for $(N-1)^{-1} ((N-1/2)^{-1})$ in (3a) ((3b)) respectively.

The model used for the hot is similar in many respects to those used by previous authors investigating ECH wave phenomena [Young et al., 1973; Ashour-Abdalla and Kennel, 1978; Hubbard and Birmingham, 1978; Curtis and Wu, 1979]. It has a pitch angle distribution which is loss-cone-like and a power law form for energy dependence. It differs from the more conventional Dory-Guest-Harris (DGH) distribution [Dory et al., 1965] in that at any fixed value of pitch angle, the spectrum is monotonically decreasing in energy while still exhibiting the positive slope feature $dE/d\theta > 0$. As far as we know, this form is perfectly reconcilable with satellite observations made at Earth and Jupiter. However, the DGH distribution displays a nonmonotonic bump in energy. Our instability analysis will then focus entirely on effects arising from the pitch angle dis-
distribution without any possible effects arising from a 'bump in energy' or 'bump on tail' aspect.

The DGH distribution is useful in other ways since it has a well-defined temperature or characteristic energy of the hot $T_h$. However, because of the rapid exponential dependence in energy, the computed growth rates tend to be too restricted in frequency. The power law form in (1) produces a broader range of unstable frequencies, a feature we believe to be more in line with wave observations. In place of the parameter $T_h$ in the DGH analysis we will refer only to a resonant energy $T_n = mV_n^2/2$ where

$$V_n = \left(\omega - n\Omega_0\right)/k_z$$

(4)

and a greater flexibility is achieved since we need refer only to the flux of resonant electrons $j_n(T_n)$ in (2) [Kennel and Petschek, 1966; Barbosa and Coroniti, 1976]. Again, the correspondence is just $T_n(\text{DGH}) \rightarrow T_n = T$.

3. INSTABILITY ANALYSIS

a. Formulation

In the limit that $n_s \ll n_e$, we may use the approximate relation

$$\gamma = -\varepsilon_i/(\partial\varepsilon_i/\partial\omega)$$

(5)
to obtain the temporal growth rate. The dielectric constant for electrostatic waves in finite temperature plasma is given as [Harris, 1959; Ashour-Abdalla and Kennel, 1978]

$$\varepsilon = 1 + \frac{4\pi e^2}{mk^2} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{dv_z G_n}{\omega - k_z v_z - n\Omega}$$

(6a)

where

$$G_n = 2\pi \int_0^{\infty} v_z dv_z J_n^2(k_z v_z/\Omega) \left[ n\Omega \frac{\partial f}{\partial v_z} + k_z^2 \frac{\partial f}{\partial v_z} \right]$$

(6b)

$$\Omega = eB_0/mc$$

and $m$ is the electron mass. If we use (1) for the hot ions, the imaginary part of the dielectric constant resulting from the superthermal contribution is

$$\varepsilon_i = \frac{16\pi e^2\omega A}{mk^2k_z} \sum_{M,N=0}^{\infty} \int_0^{\infty} dv_z \frac{V^{2M-1}J_z(k_z v_z/\Omega)}{(V_n^2 + v_z^2)^{N+1}}$$

$$\left[ \frac{(M + N + 1)v_z^2}{(V_n^2 + v_z^2)^{N+1}} - \frac{Mn}{\omega} \right]$$

(7)

where $\omega = \omega/\Omega > 0$ and $A = A(2M)^{N+1}$. It will become clear later than negative values of $\varepsilon_i$ correspond to wave growth, $\gamma > 0$, for $k_z > 0$ and arise from the second term when $M > 0$. The evaluation of the integrals in (7) for integer values of $M$ and $N$ is straightforward, but cumbersome. We will confine our analysis to the specific value $M = 1$ in this paper corresponding to an anisotropy of 1 as defined by Kennel and Petschek [1966]. Further development of (7) is included in the appendix and it may be shown that (7) is alternately written as

$$\varepsilon_i = \frac{8\pi e^2\omega^2}{n_e a} T_j(T_n) \sum_{n=0}^{\infty} \frac{T_n^{H-1}}{T_n} (2\mathcal{J}_n)^{2}\omega^2 \int_{-\infty}^{\infty} dv_z \frac{1}{k_z^2} \frac{\mathcal{J}_n(N,\beta,\bar{\omega})}{(\bar{\omega} - n)}$$

(8)

Here $\omega^2 = \omega_0^2/\Omega^2$, $a^2 = 2T_i/m$, $d_z^2 = k_z^2 a^2/2\omega z$, and $\mathcal{J}_n$ is defined in (A6). We have approximated $k^2 = k_z^2$ in (7) appropriate for electrostatic wave propagation quasi-perpendicular to the magnetic field. An important parameter that arises in the analysis is the quantity

$$\beta = |k_z V_n/\Omega| = |k_z(\bar{\omega} - n)/k_z|$$

(9)

which for stimulated emission of waves by superthermal electrons takes on values $\beta > 1$. As shown in the appendix the function $\mathcal{J}_n$ is then described by modified Bessel functions in their asymptotic form. The parameter $\beta$ is analogous to the quantity $k_z\rho_n$ which appears when the DGH distribution is used for the hots.

b. Wave Properties

In a regime where $n_s \ll n_e$, the wave dispersion properties are determined by the real part of the dielectric constant and papers relevant to our discussion here are Bernstein [1958], McAfee [1969], Oya [1971], Gaffey and LaQuey [1976], Rønmark et al. [1978], Curtis and Wu [1979], and Barbosa [1980]. For our purposes here we employ the procedure used by Barbosa [1976, 1980]. The real part of the dielectric constant for electrostatic waves in finite temperature isotropic Maxwellian plasma is expanded in powers of $(k_z/k_c)^2$

$$\varepsilon = \varepsilon_0 + \varepsilon_1(k_z/k_c)^2 + 0(k_z)^4 = 0$$

(10)

where

$$k_z = (\omega - n\Omega)/a$$

(11)

Formulas for $\varepsilon_0$ and $\varepsilon_1$ can be found in McAfee [1969] for the pure electrostatic approximation and a number of solutions of (10) are given in Barbosa [1980] for the case of $\omega_z^2/\Omega^2 = 16$. Larger values of this parameter are realized in Jupiter's magnetosphere but in the limit $\omega_z^2 >> 1$ the lower harmonic band solutions become independent of $\omega_z$. The solutions in Figure 1 of Barbosa [1980] are displayed in terms of the variable $d = k_z/(2\omega_z)$ which is the index of refraction normalized to the electron thermal speed rather than light speed in vacuum. We shall concentrate on the first, second, and upper hybrid bands in this paper. For sufficiently large values of $k_z$, cyclotron damping on the cool background electrons will occur and we have taken the arbitrary condition that

$$|k_z/k_c| = |a/\omega| \leq 1$$

(12)

must be satisfied in order to neglect cyclotron damping on the background. In the limit $k_z \rightarrow 0$, (10) reduces to

$$\epsilon_r = 1 + \frac{2\omega_c^2}{k_z^2 a^2} \left[ 1 - \sum_{n=0}^{\infty} e^{-b_+^2 I_0(b_+^2)} \frac{\omega}{\omega_n - n} \right]$$

(13)

where $b_+ = \omega_c d_+$ and we have immediately that

$$\partial \varepsilon_r / \partial \omega_0 = \frac{\omega_c^2}{b_+^2} \sum_{n=0}^{\infty} e^{-b_+^2 I_0(b_+^2)} \frac{4\omega_c^2}{(\omega^2 - n^2)^2} > 0$$

(14)

When considering frequencies near the upper hybrid, $\omega_U = (1 + \omega_c^2)^{1/2}$, the expansion of (13) for small values of $b_+$ is useful and is given as

$$\epsilon_r = 1 + \frac{\omega_c^2}{1 - \omega_U^2} \left[ 1 - \frac{3b_+^2}{4 - \omega_U^2} + 0(b_+^2) \right]$$

(15)

We have then the approximate relation near $\omega = \omega_U$

$$\partial \varepsilon_r / \partial \omega_0 = 2(1 + \omega_c^2)^{1/2}/\omega_c^2$$

(16)
Fig. 1. Maximal temporal growth rates for the first (solid line) and second (dashed line) harmonic bands for the threshold given by (17). The frequency within each band is $\omega = n + \eta$. Increasing the threshold to the value 4 (equivalent to decreasing $k_z$) results in the slashed line modification to the first harmonic growth rate. A similar effect occurs also for the second harmonic.

c. Temporal Growth Rates

In order to simplify the use of (5) we shall use (14) which was derived assuming $k_z = 0$. The error involved can be seen from (10) to be $O(a^2/V_n^2)$ which is only 11% for $|V_n/a| = 3$. This is actually a constraint we shall consider for all our growth rates. The condition (12), we recall, indicates when cyclotron damping on the isotropic background electrons begins. If we take $|V_n/a| = 3$ as a limit we are sure to be free from damping effects and sufficiently superthermal so as to invoke a distribution like (1). Thus, the maximum value of $k_z$ allowed in our analysis is such that

$$|V_n/a|_{\text{max}} = 3 \quad (17)$$

for any value of the index $n$. The condition (17) then constrains $\beta$ to values

$$\beta \geq (18)^{1/2} \bar{\omega} d_\perp \quad (18)$$

where the values of the right-hand side are fixed by the solution to (10).

With the constraint given by (18), the growth rate may be obtained numerically by direct evaluation of (8) and (14). We have plotted the result in Figure 1 for the parameters $N = 1$, $\bar{\omega}_e^2 = 16$, $n_e = 100 \text{ cm}^{-3}$, $T_e = 100 \text{ eV}$, and $T_{j_b} = 10^4 \text{ cm}^{-2} \text{ s}^{-1} \text{ sr}^{-1}$. Under conditions where $\bar{\omega}_e^2 \gg 1$, the growth rate in the lower harmonic bands becomes independent of $\bar{\omega}_e^2$ and Figure 1 may be scaled explicitly as

$$\frac{\gamma}{\Omega} \sim \frac{T_{j_b}(T)}{n_e a} \quad (19)$$

for other values of these parameters. The solid (dashed) lines correspond to growth in the first (second) harmonic bands.

Fig. 2. Maximal convective growth rates normalized to the background electron thermal speed. The smaller relative group velocities for the lower frequency wing (i.e., (22)) results in a bias towards lower frequencies within the band.
The slashed line shows what happens to the first harmonic band growth rate if we raise the constraint (17) to $\left| V_{n}/a_{\text{min}} \right| = 4$. It reduces the growth rate by a factor $\left( \frac{3}{4} \right)^{4}$ over the lower part of the frequency band where the constraint (18) is enforced at larger perpendicular wavenumbers. A similar effect occurs for the second band. The higher frequency portion, $\eta > 0.5$, is unaffected since instability is occurring there due to higher energy electrons than specified by (17). We can, therefore, identify the lower ($\eta < 0.5$) spans of both harmonic bands as due to superthermal electrons with velocities in the vicinity (17) and the higher ($\eta > 0.5$) spans due to significantly higher velocity electrons. Since the spectral index $N = 1$ encompasses the possibility that large fluxes exist over a broad energy range of particles, the existence of waves in the higher span is, in this model, associated with flat electron spectra extending to higher energies than (17). For instance, the stability at $\tilde{\omega} = 1.8$ in Figure 1 is due to electrons with $|V_{n}/a| = 6.4$. If the spectrum is actually steeper, $N > 1$, we may observe that in (8) the factor $(T/T_{e})^{N-1}$ will substantially reduce the relative growth rate in the higher span ($\eta > 0.5$) compared to the lower span where (17) is operative. In summary, flat electron spectra imply significant growth at both frequency wings centered on $(n + 1/2)\omega_{ce}$, whereas steeper electron spectra result in a bias toward the lower span $\eta < 0.5$.

Another notable feature of Figure 1 is the growth rates in both harmonic bands are comparable. When we compute the convective amplification, it will be seen how the first band actually dominates.

We may also estimate the temporal growth rate in the vicinity of the upper hybrid frequency. This is not meant to be an exhaustive treatment of this band but it is important to see how this band fits quantitatively with our results for the lower harmonic bands. We hope to investigate this band more thoroughly at a later date. For the present we consider still $\tilde{\omega}_{n} = 16$ and, referring either to Figure 1c of Barbosa [1980] or (15) of this paper, we find that for small $d_{i}$

$$\tilde{\omega} \approx (1 + \tilde{\omega}_{n}^{2})^{1/2} \left[ 1 + \frac{3\tilde{\omega}_{n}^{2} d_{i}^{2}}{2(\tilde{\omega}_{n}^{2} - 3)} \right]^{1/2} \quad (20)$$

and $\tilde{\omega}(d_{i} = 0.1) = 4.2$. Evaluation of (8) along with (16) provides the estimate $\gamma/\Omega \approx 2.7 \times 10^{-4}$ at $\tilde{\omega} = 4.2$ which exceeds the growth rates in Figure 1 by a factor of $\approx 4$. At this frequency the resonant electrons responsible for the growth are $|V_{n}/a| \approx 15.2$. Thus, the narrowband emission near the upper hybrid is seen to be more intense than the lower harmonic bands and associated with very energetic electrons. This result is consistent with that of Kurth et al. [1979] and fits in perfectly with the requirement based on phenomenological grounds that the cause of very intense upper hybrid noise must be tied to 'the singular convective properties of the upper hybrid band itself' [Barbosa, 1980].

d. Convective Amplification

Ashour-Abdalla and Kennel [1978] have shown attention to the importance of considering the convective growth rate. The relationship is simply

$$k_{c} = -\gamma/|V_{n}| \quad (21)$$

For the first and second harmonic bands we may make use of an approximate power law relationship between $\omega$ and $k_{c}$ demonstrated in section 4 for a limited range of frequencies to obtain

$$V_{n}/a = \tilde{\omega}^{4.27} \quad (22a)$$

for the first band and

$$V_{n}/a = \tilde{\omega}^{1.65} / 727 \quad (22b)$$

for the second band. The results are shown in Figure 2. The smaller relative group velocity for frequencies closer to the lower end of the band has introduced another bias towards the lower span. The effect of increasing the threshold $|V_{n}/a|_{\text{min}}$ is also shown to 'peel off' the growth rate at lower frequencies leading to a redistribution of growth favoring middle frequencies. This is a very interesting effect and we explore it in greater detail in Figure 3 for the first harmonic band. The condition (17) was chosen arbitrarily as a threshold for the transition to the superthermal loss cone distribution. For a given frequency it is a statement about $k_{c}$. We have relaxed the condition to include values $|V_{n}/a|_{\text{min}} = 2-5$. There is a very dramatic shift towards low frequencies $\tilde{\omega} = 1.1-1.2$ when the threshold is lowered. This result is deemed to have a direct correspondence with spectral observations of ECH banded emissions which sometimes indicate (1) a narrowband emission near $\tilde{\omega} = 1.2$ [Hubbard and Birmingham, 1978]; (2) a broader emission from $\tilde{\omega} = 1.1-1.6$ or higher (see, e.g., Figure 2 of Kurth et al. [1980]); and (3) a spectral peak near $\tilde{\omega} = 1.6$.

\[ \text{Fig. 3. Maximal convective growth rate for the first harmonic band for various thresholds. This figure indicates that the maximum growth rate (with respect to } k_{c} \text{) for a given frequency occurs at a smaller } V_{n} = (\omega - n\Omega)/k_{c} \text{ for lower frequencies. The maximum growth rate at correspondingly lower frequencies is thus sensitive to the value of the arbitrarily imposed threshold which serves to delineate the transition from a loss cone distribution to a "filled-in" isotropic distribution [Ashour-Abdalla and Kennel, 1978].} \]
or 1.7. In terms of this model, this variability of observed spectral features can be accounted for by just the simple device of raising the resonant electron threshold. In fact, one consequence of this interpretation is that the wave spectrum can evolve dynamically to higher thresholds. If we start out with a particle distribution which produces growth according to \( |V_r/a| = 2.5 \) as in Figure 3, the back reaction of waves on the particle distribution can remove the loss cone feature at low energies producing a threshold and a corresponding wave spectrum that evolves through the sequence \( |V_r/a|_{\text{thresh}} = 2.5 \rightarrow 5 \) and higher assuming no electron sources operate on this evolutionary time scale to rejuvenate the loss cone fluxes at low superthermal energies. We hope to test this suggestion by examination of extensive wave data taken by the International Sun-Earth Explorer (Isee) satellite. At the present, this idea of dynamical evolution of spectra is only conjecture but seems promising.

The upper hybrid convective growth rate may be estimated by considering that for small \( d_i \) the perpendicular group velocity may be computed from (20) as

\[
V_g/a \simeq \frac{3\omega_p^2d_i}{(2)^{3/2}(\omega_p^2 - 3)}
\]

(upper hybrid) and the estimate for its convective growth rate is \( k\alpha/\Omega \approx 1.0 \times 10^{-3} \) which shows that the upper hybrid still dominates provided that very energetic electrons are present. We shall summarize the results of this section in section 6 where we discuss the implications of this model.

At Jupiter the first harmonic band has been observed as a durable feature of the inner magnetosphere [Kurth et al., 1980]. We would like to have a general scaling law which then keys on this particular band for latitudes very close to the magnetic equator. If a wave propagates a distance \( \delta r \) across a dipole magnetic field near the equator, the corresponding change in normalized frequency is given by

\[
\delta r/\rho = \delta\omega/3\omega
\]

We make the crude approximation that in Figure 2 the convective growth rate for \( |V_r/a| = 3 \) is a constant (\( k\alpha/\Omega \approx 1.5 \times 10^{-4} \)) over the range \( \Delta \omega = 1.1-1.6 \) for the first harmonic and zero elsewhere in the band. If we then define a critical flux of resonant electrons, \( T_j^*(T) \), as that able to produce 10 e-foldings in amplitude over a distance such that \( \delta \omega = 0.1 \), we find using (19), (22) and (24) that at Jupiter

\[
T_j^*(T) = 23 \left( \frac{0.1}{\delta \omega} \right) pR^2\omega \text{ (cm}^2 \text{ s sr)}^{-1}
\]

where \( R \) is in units of planetary radii \( R_j \) and the cool background pressure \( p = n_cT_c \) is in units of eV/cm\(^3\). The numerical coefficient, we recall, is computed for the constraint (17) and we may also write, for \( T = T_j \) as

\[
T_j^* = 23 \left( \frac{0.1}{\delta \omega} \right) \left[ \frac{4.2G}{B_j} \left( \frac{7.13 \times 10^4 \text{ km}}{R_j} \right) \left( \frac{T/T_j}{9} \right) pR^2\omega \right]
\]

thereby allowing for smaller thresholds, as in Figure 3, and other planetary magnetospheres. Equations (25) and (26) are a
major result of our paper. Fluxes much below the values in (25) are not likely to produce conspicuous wave intensities unless special or preferred amplification ray paths exist so as to increase the effective gain. Within factors of 0(1), (25) defines a flux level at which wave generation in the 3/2's band should be readily obtained and quite conspicuous observationally. There is an explicit dependence on distance \( R \) and background pressure indicating preference for wave growth in general (1) closer to the planet and (2) in low pressure plasmas. We note also that the frequency dependence in (25) indicates that the first band should dominate over the higher bands since the convective growth rates are comparable. This results primarily from the fact that according to (24) successive higher bands are more sensitive to convective changes \( \delta \omega \) and the total amplification lengths are smaller relative to the 3/2's band.

4. WAVE PROPAGATION

a. In the Magnetic Equatorial Plane

As stated earlier, for \( \alpha \omega_b^2 = \omega_c^2 / \Omega_e^2 \gg 1 \) the convective properties of the lower ECH bands depend only on the magnetic field \( B \) whereas the higher bands near and above the upper hybrid depend on both \( \delta \omega_p \) and \( \delta \Omega_e \) [Barbosa, 1980]. We will address the problem of wave propagation for the lower bands in the high \( \omega_c \) limit.

We consider for simplicity waves travelling exactly in the magnetic equatorial plane of a dipolar field. Excursions in magnetic latitude are treated in section 4b. The problem reduces to propagation in an isotropic, inhomogeneous spherically stratified medium. Kelso [1964] has given the solution to this problem. If \( \theta \) is the angle between the wavevector and the radial vector \( r \) from the magnetic dipole, the ray paths are such that the following relation holds (Snell's law in spherically stratified medium):

\[
n r \sin \theta = \text{const} \quad (27)
\]

where \( n = n(r) \) is the wave index of refraction and \( r \) is the radial distance to dipole center. The solutions to the dispersion equation for perpendicularly propagating Bernstein waves are well known. For the case \( \alpha \omega_b^2 = 16 \), the solutions for the lowest two bands are shown in Figure 4 as circles and squares. We have fitted a power law curve through the circle points. As can be seen, the fit does reasonable justice to the solution but becomes very poor very close to the cyclotron harmonics. We use this power law fit as a convenient representation, and if \( n(\omega) \sim \omega^{-\alpha} \) then as a function of radial distance in a magnetic dipole field

\[
n(r) \sim r^{-3\alpha} \quad (28)
\]

These lowest two bands below the upper hybrid frequency are actually backwards travelling waves; however, we incur no error here if we treat them as forward propagating waves since the medium is isotropic. If \( \theta_0 \) and \( r_0 \) are referred to the initial launch conditions and \( \phi \) is the position azimuth, then since

\[
\tan \theta = \frac{\sin \theta_0}{\cos \theta_0} \quad (29)
\]

by geometrical arguments we may rewrite (29) and integrate to obtain

\[
\int_{\phi_0}^{\phi} d\phi = \int_{r_0}^{r} \frac{d\phi}{\sin \theta_0} \frac{n \Delta \theta}{(n^2 - n_0^2 \sin^2 \theta_0)^{1/2}} \quad (30)
\]

We take \( \phi_0 = 0 \). Upon employing the relation (28) we find that the integral in (30) is elementary and the solution for the ray path is given finally as

\[
r = \frac{\sin [(3\alpha - 1)\phi + \theta_0]}{\sin \theta_0} \quad (31)
\]

The solution (31) is shown in Figures 5 and 6 for the first and second harmonic bands for various initial launch (reception) angles. All distances, henceforth, are normalized to the radial distance of launch (observation). Notable features are of a reflection point at

\[
r_{\text{fl}} = (\csc \theta_0)^{1/(3\alpha - 1)} \quad (32)
\]

occurring at

\[
\phi_{\text{fl}} = \frac{1}{(3\alpha - 1)} \left( \frac{\pi}{2} - \theta_0 \right) \quad (33)
\]

The azimuthal 'range' is \( \phi_{\text{fl}} = 2\phi_{\text{fl}} \) when \( r = 1 \) again. We have also indicated radial distances at which the normalized frequency changes maximally for propagation in a particular band. For instance, in Figure 6 a wave launched with \( \theta_0 = 5^\circ \), \( \phi = 2 \) will evolve to \( \phi = 3 \) upon interception of the arc labelled (3/2)\( ^{1/3} \). Note also the relative compression of ray paths and smaller accessible radial increments for propagation within a band for the higher frequency bands (i.e., (24)).

b. In the Magnetic Meridian Plane

The convective growth rate is a sensitive function of \( V_p/a \) and any calculation of the amplification should take careful account of changes in \( k_r \) due to wave propagation. We con-

Fig. 6. Same as Figure 5 for the second harmonic band.
centrate here on the lower ECH bands and show index of refraction surfaces for the first harmonic band in Figure 7. This is an expanded version of Figure 1a of Barbosa [1980] to include the lower frequencies. The dotted lines indicate where (12) is violated and heavy damping on the background is expected to occur; the solid lines show where superthermal interaction occurs.

We wish to establish here ray paths for propagation in the magnetic meridian plane by the method of Poeverlein. Figure 8 shows a sketch of various geometrical angles required with reference to a coordinate system oriented with the z axis along the magnetic field, as in Figure 7. If Λ is the magnetic latitude, the angle ξ is given by

\[ ξ = \sin^{-1} \left( \frac{2 \sin Λ}{1 + 3 \sin^2 Λ} \right)^{1/2} \approx 2Λ \tag{34} \]

and the angle ξ by

\[ ξ = \tan^{-1} \left( \frac{3 + 5 \sin^2 Λ}{1 + \sin^2 Λ} \tan Λ \right) \approx 3Λ \tag{35} \]

The approximate forms of (34) and (35) are obtained for small latitudes.

Referring back to Figure 7, we note that the group velocity is normal to the refraction surfaces in the direction of increasing ω for any value of (kz, kx) in the diagram. Equation (34) then determines whether a ray is climbing or dropping in magnetic latitude during propagation. Application of Snell's law requires that the component of k perpendicular to \(\nabla B\) be conserved during propagation (again, in the high density limit the dispersion properties are independent of \(\nabla ω\), for the lower ECH bands). We have superimposed the direction of \(\nabla B\) in Figure 7 using (35) for latitudes of 1°, 2°, and 3°, and the curves are shown emanating from a point on the \(\omega = 1.5\) surface. For other values of k, the curves may be translated preserving their orientation with respect to Bo. A wave launched with this value of k will then migrate to a neighboring point along a \(\nabla B\) line computed at the instantaneous magnetic latitude. Since (35) shows a strong dependence on latitude, we see that even only 3° off the equator propagation tends to significantly alter the value of k, and thereby the coupling to the hot electron distribution.

To show this in more detail, we have applied this construction at increments of \(δ\omega = 0.05\) for the point under consideration. To compute the change in latitude at each step we use the approximate formula

\[ δΛ ≈ \tan^{-1} \left( \frac{δ\omega}{3\omega \cot Ψ} \right) \tag{36} \]

where \(\cos Ψ = V_p B_0 / V_e B_0\). Equation (36) was derived assuming the magnetic induction has no latitudinal dependence \(|B| \sim r^{-3}\) appropriate near the magnetic equator. The construction is shown in Figure 7 for a wave \(\omega = 1.5\), launched from the equator Earthward and into the southern hemisphere. At the point where \(\omega = 1.25\) the latitude is \(-4.4°\) and very shortly
Fig. 9. Radial profiles of electron density and effective plasma temperature in Jupiter's magnetosphere. (a) The circles are equatorial values obtained from Warwick et al. [1979] and the square is determined from the Plasma Wave Experiment. A break occurs in the profile at ~20 R_J where the plasmadisc forms on the nightside. (b) Plasma temperature determined from (37) and the density profile. See the text for discussion of how this figure relates to thermal and superthermal electron energies also.

after that the wave group velocity is redirected towards smaller latitudes. Similar constructions can be made for other initial k vectors launched from the equator with basically the same effect. For initial launch away from the Earth, the modification to the diagram consists of \( \nabla B \) lines tilted in the opposite sense and the effect of latitudinal reflection and trapping can also occur. We recall that the criterion for significant wave amplification, (26), was written to encompass a convective change of \( \delta \omega = 0.1 \) at constant \( |V_E/a| = 3 \) and we see that this is readily accomplished in the construction shown in Figure 7. While this is in general easily managed at the equator we note again that waves launched from as little as 3° from the equator experience a rapid change in resonant velocity and the total path-integrated gain can be expected to be significantly reduced over that obtainable at the equator [Barbosa, 1980]. Thus, two important aspects of this construction are borne out: (1) waves off the equator are not locally amplified as effectively as they are at the equator, and (2) waves that are amplified at the equator cannot propagate freely away from there due to the trapping effect and be observed as intense noise (as can, for example, whistler mode noise). This result is compatible with the observations of sharply defined statistical occurrence regions near the magnetic equator [Christiansen et al., 1978; Gough et al., 1979; Kurth et al., 1980].

5. APPLICATION TO JUPITER

Having described some of the features of the theory, we now turn our attention to how these results may be applied to Jupiter's inner magnetosphere. We have already demonstrated how propagation even slightly off the magnetic equator affects the value of \( k \), and therefore the resonance energy. This result compares very well with Voyager observations of
Fig. 10. Critical flux of resonant electrons required to excite the lowest gyroharmonic band. The diamonds are 130–183 keV electron fluxes multiplied by 0.13 MeV obtained from Krimigis et al. [1979] for three equatorial crossings of Voyager 1. The theoretical curve for $T_e = T_i$ are shown as solid lines for various models of plasma pressure (see text). The theoretical curve for the model given by (37) for ion pressure and $T_i = 10T_e$ is shown as the slashed line.

Two cases may be considered now for evaluation of (25): $T_i = T_e$. If the ion and electron thermal temperatures are equal, we may substitute (37) directly into (25). We let $\tilde{\omega} = 1.5$ for the remainder of the discussion. The result is shown in Figure 10 as the curve labelled $p \sim R^{-3.54}$. To assess this model we have plotted observed fluxes of 130–183 keV electrons taken from Krimigis et al. [1979] for three equator crossings of Voyager 1. On the assumption that the spectrum is flat, $N = 1$, $j_{ij}$ is then the same value for energies extending down to $T_i/T_e = 9$. It is apparent that the theoretical curve lies too far above the observed fluxes. If we take instead a constant temperature model (50 eV), the result is also shown in Figure 10 behaving as $R^{-3.57}$. This model is seen to be unsatisfactory also. Two hybrid models between these two extremes are shown with the plasma pressure varying as $p \sim R^{-3}$, $R^{-2}$ up to the point where the temperature drops to 50 eV and continues into 6 $R_J$ at constant temperature. While the $R^{-3}$ model does mediate the extremes, it is still unsatisfactory; considering that it is ad hoc it should do better. We abandon the case $T_i = T_e$ forthwith.

$T_i > T_e$. Inspection of Figure 10 at this stage yields a very simple and natural model to consider. If $T_i > T_e$ and we still extrapolate the plasma pressure profile (37) into $R = 6$, a reasonable agreement can be obtained if $T_i \sim 10T_e$. Further details of this model are that the electron temperature at 6 $R_J$ is $\approx 20$ eV which compares favorably with EUV observations in the plasma torus [Broadfoot et al., 1979] while at 20 $R_J$ the electron temperature is $\approx 2$ keV consistent with the results of Bridge et al. [1979]. This model would appear to reconcile the results of five independent experiments over this radial range, and considering its relative simplicity should be regarded as an excellent candidate.

The electron temperature is displayed, therefore, in Figure 9b with an appropriate rescaling of the ordinate down by a factor of 10. On the assumption that superthermal electrons with energies $T_i/T_e = 9 \approx 10$ drive the lowest gyroharmonic
band emissions, the resonant electron energy profile is also shown in Figure 9b scaled as is. We see that a loss cone distribution of 0.2 keV (at $R \approx 6.0 R_J$) to 20 keV (at $R \approx 20 R_J$) superthermal electrons is required for instability of the 3/2's band with fluxes on the order of $T_{\parallel} \sim 3 \times 10^7$ (cm$^3$ s)$^{-1}$.

Whether a loss cone actually exists at these energies over such a large range of distance requires further examination especially at the peak of the high density plasma torus at $6 R_J$. In Appendix B we give a criterion for a lower bound to $|V_{\parallel}|/\tau_{\text{min}}$ assuming the total electron distribution is simply the sum of an isotropic background Maxwellian and a loss cone superthermal distribution (2). It may then be verified that the models for this section do satisfy $n_e/n_i \ll 1$. We consider here the aspect of electron pitch angle scattering due to binary coulomb interactions [Evitaer et al., 1979]. For superthermal electron velocities $v$ we use the formula given by Kroll and Trinity [1973]:

$$\tau_\rho \propto \frac{m^2 v^3}{16 \pi e^4 \ln \Lambda} \quad (38)$$

for scattering off of thermal electrons. We take $\ln \Lambda = 26$ and use the models given in Figure 9 with $T_i = 10 T_{\text{e}}$. The pitch angle relaxation time is then scaled as

$$\tau_\rho \sim 1 \text{ day} \left( \frac{T/T_{\text{e}}}{9} \right)^{1.1} \left( \frac{R}{8} \right)^{1.1} \quad (39)$$

for the inner-middle magnetosphere. For distances $R \approx 8 R_J$ it seems possible that loss cone features can be produced at these low superthermal energies, but any closer to Io it appears that both the loss cone and the flux magnitude could be tailored according to (39) or any other source/sink processes.

As a final consistency check, we consider how VLF chorus emissions at $\omega \approx f_{\text{ch}}/2$ fit into this picture. Voyager 1 observed an intense emission of this type near $\approx 8 R_J$ on the inbound leg and Coroniti et al. [1980] have analyzed the event. If we take a resonant electron energy appropriate for emission at exactly $f = f_{\text{ch}}/2$, then using the density model in Figure 9b, a dipole magnetic field of 4.2G at the equator, and the expression $T_\rho = mc^2/8\omega$ gives, we obtain

$$T_\rho \approx 1 \text{ keV} \left( \frac{R}{8} \right)^{1.37} \quad (40)$$

(chorus). Coroniti et al. [1980] thus concluded that significant fluxes of few keV electrons exist within the Io plasma torus. Comparing (40) to the resonant electron energies at $T/T_{\text{e}} \approx 10$ responsible for 3/2's emissions in Figure 9b, we note that at $8 R_J \sim 650$ eV electrons are required. Indeed, at this distance both chorus and Bernstein modes were detected consistent with the requirement that the same source electrons drive both modes when the interaction energies are comparable. Referring to Figure 4 of Kurth et al. [1980], we note also that from $6-8 R_J$ the frequency channels just below the electron gyrofrequency do not show much evidence for wave activity due to chorus or hiss at frequencies greater than 10 kHz. This is also consistent with the suggestion that electrons at low superthermal energies are not unstable to whistlers either unless the resonant energies are sufficiently large ($\approx 10$ kHz) to avoid isotropization. At $6 R_J$, (39) gives $T \approx 2.2$ keV for 1 day relaxation time and this is well above the resonant energy for chorus (given by (40)) or the 3/2's band (given by Figure 9b).

6. SUMMARY AND DISCUSSION

The theory developed here is based on a low density superthermal electron population which has a loss cone pitch angle distribution and is unstable to Bernstein modes of the background electrons. In the case that $\omega_e/\Omega_e > 1$, lower harmonic bands can be excited by superthermals with energies ranging from $T/T_{\text{e}} = 4-50$ above background electron temperature. Based on just consideration of the convective growth rate, features in the wave spectrum can be associated directly with certain resonant energy electrons. A loss cone which extends as far in as $T = 4T_{\text{e}}$ results in a strong emphasis on growth of waves at the lower frequency part of the band ($\omega \approx 1.2$). Filling in the loss cone at these lower energies and raising the threshold to $T = 50T_{\text{e}}$ result in a reemphasis favoring the higher frequency part of the band ($\omega \approx 1.8$). Intermediate values of the threshold mediate these two extremes permitting a peak in the wave spectrum to occur in the range $\omega \approx 1.2-1.8$.

The power law energy dependence of the model electron distribution has an important function in this analysis. It provides significant superthermal electron fluxes over as large a range of energies discussed and results in computed growth rates which are positive (unstable) over the major portion of the harmonic frequency bands. Thus, we can account for spectra which may show not only a narrow peak at $\omega \approx 1.5$ but also weaker emission throughout the band. The flat energy spectrum of the model also allows for the possibility of very energetic electrons $T > 100T_{\text{e}}$ which we have suggested are responsible for intense emission near the upper hybrid frequency. Whether an intense emission in the upper hybrid band occurs coincident with intense lower harmonic bands can be tied to whether or not the responsible electrons are present at the required energies and are sufficiently unstable. Thus, we can account for another variability of banded ECH wave phenomena which may show various combinations of harmonic bands present (e.g., upper hybrid alone, lower bands alone, upper hybrid plus lower bands, etc.).

For application to Jupiter, we have concentrated on the 3/2's band which was observed at each crossing of the magnetic equator for $R < 23 R_J$. The requirement that 10 e-foldings in amplitude of this convectively unstable mode be produced has yielded an approximate relation for the critical flux of resonant electrons $j_\rho$, in terms of the background thermal electron pressure. The simplest explanation for this is that to achieve...
the same amplification, a higher density background medium requires a higher flux of driving electrons in proportion and in a higher temperature medium the waves propagate that much faster. Based on observations of 130–180 keV electrons made by Voyager 1 we have found that the electron temperature is likely to be less than the ion temperature in the inner-middle magnetosphere, \(T_e \sim 10^7\). This conclusion is model dependent and is based on our assumption that the superthermal electron spectrum is flat \(N = 1\) and that the pressure profile in the nightside plasma sheet can be extrapolated into \(R = 6\). However, the compatibility of our theoretical model with in situ plasma and EUV observations lends support to at least the softer conclusion that \(T_e > T_i\). The ion/electron temperature ratio can also be assumed to be a function of radial distance. Figure 10 indicates that an ion pressure profile that varies as \(R^{-2}\) comes close to giving an ion temperature of 50 eV at 6 \(R_e\) in agreement with observations, and the electron pressure profile (slashed line) best ‘fits’ the Krimigis et al. [1979] data. These considerations then suggest that \(T_e/T_i\) increases with radial distance having the value \(T_i \approx 2.5T_e\) at 8 \(R_e\) and increasing to \(T_i \approx 10^7 T_e\) at 20 \(R_e\). When more accurate plasma and electron pressure profiles become available for this region of the magnetosphere we can then refine our theoretical model along these lines and also address questions regarding day/night asymmetries in the plasma wave data and in other Voyager data. Inasmuch as the low density superthermal model we have used is applicable to the Earth, we are aware that the analysis contained in this paper is quite compatible with terrestrial observations also and, in particular, the critical flux (26) when scaled to conditions at the Earth agrees with the empirical results of Maeda et al. [1976].

Future work in this area encompasses at least the following areas. While we have demonstrated that the magnetic equator is a preferred location for wave growth, actual path-integrated growth calculations are needed precisely of the type performed by Maggs [1976, 1978]. It is not yet clear to us from inspection of Figures 5, 6, and 7 whether any departure from gyrotrropic conditions will result. A numerical code which amplifies waves along their ray paths would indicate whether perpendicular wavevectors are received preferentially along the radial vector or in the azimuthal direction, and also would delineate the role other convective effects have in shaping the final observed power spectrum. Observationally, this question can be addressed by the Isee satellite which can determine statistically (or perhaps in a single event) the occurrence function \(I, K, I', K'\). To illustrate their basic behavior we have plotted in Figure 11 values for \(n = 1\). Since we are interested in values \(\beta > 1\), we may apply (A3) and (A4) to the asymptotic forms for the Bessel functions. If \(\mu = 4n^2\) we have for \(c = 1\)

\[
F_{1} \sim \frac{1}{2} \left[ 1 - \frac{1}{2} \left( \frac{\mu - 1}{(2)2^2} + \frac{1}{2} \frac{\mu - 1(\mu - 9)}{(2)2^4} + \ldots \right) \right]
\]

\[
F_{2} \sim \frac{1}{2} \left[ 1 - \frac{1}{2} \left( \frac{\mu - 1}{(2)2^2} + \frac{3}{2} \frac{\mu - 1(\mu - 9)}{(2)2^4} + \ldots \right) \right]
\]

\[
F_{3} \sim \frac{1}{2} \left[ 1 - \frac{1}{2} \left( \frac{\mu - 1}{(2)2^2} + \frac{5}{2} \frac{\mu - 1(\mu - 9)}{(2)2^4} + \ldots \right) \right]
\]

\[
F_{4} \sim \frac{1}{2} \left[ 1 - \frac{1}{2} \left( \frac{\mu - 1}{(2)2^2} + \frac{7}{2} \frac{\mu - 1(\mu - 9)}{(2)2^4} + \ldots \right) \right]
\]

etc. These expressions are quite accurate for values of \(z = \beta\) to the right of the maxima. We may continue by forming differences appropriate for (A1) and defining

\[
\mathcal{A}(N, \beta, \omega) = (N + 2) \int_{0}^{\infty} \frac{x dx J_{\beta}^2(x)}{(1 + x^2)^{N+2}} \left[ \frac{x^2}{1 + x^2} - \frac{n\Omega}{N + 2} \right]
\]
we have finally
\[ \mathcal{A}(N=1) \sim \frac{1}{2} \frac{3}{4} \frac{1}{\beta} \left[ \frac{1}{\omega} \left( 2n - \frac{\mu - 1}{\omega} \right) \right] + \frac{5}{2} \frac{(\mu - 1)}{(2\beta)^2} \left[ 1 + \frac{2n}{\omega} \right] 
- \frac{5}{7} \left( \mu - 1 \right) \frac{1}{\omega} \left( 2\beta \right) \left[ 2 + \frac{2n}{\omega} \right] \}
\]
\[ \mathcal{A}(N=2) \sim \frac{1}{2} \frac{3}{4} \frac{1}{\beta} \left[ \frac{1}{\omega} \left( 2n - \frac{\mu - 1}{\omega} \right) \right] + \frac{5}{2} \frac{(\mu - 1)}{(2\beta)^2} \left[ 1 + \frac{2n}{\omega} \right] 
- \frac{7}{9} \left( \mu - 1 \right) \frac{1}{\omega} \left( 2\beta \right) \left[ 3 + \frac{2n}{\omega} \right] \}
\]
\[ \mathcal{A}(N=3) \sim \frac{1}{2} \frac{3}{4} \frac{1}{\beta} \left[ \frac{1}{\omega} \left( 2n - \frac{\mu - 1}{\omega} \right) \right] + \frac{5}{2} \frac{(\mu - 1)}{(2\beta)^2} \left[ 1 + \frac{2n}{\omega} \right] 
- \frac{9}{11} \frac{1}{\omega} \left( 2\beta \right) \left[ 3 + \frac{2n}{\omega} \right] \}
\]
such that
\[ \epsilon_i = \frac{16\pi^2 e^2 \omega}{m e k^2} \sum_{n=\infty} \int_{0}^{\infty} V_n^{-2(n+1)} \mathcal{A}(N, \beta, \omega) \quad (A8) \]

APPENDIX B

Since the results in Figure 3 depend sensitively on the value of $[V_e/a_{\text{min}}]$, it is important to have a quantitative estimate of this parameter. We can obtain a least lower bound for $[V_e/a_{\text{min}}]$ by assuming the background Maxwellian is isotropic and only the superthermal distribution (2) has a loss cone. The criterion is just that the integral number densities are comparable at $[V_e/a_{\text{min}}]$. For an isotropic Maxwellian, the integral number density is
\[ N(>x) = 4n_e x^{1/2} \int_{x}^{\infty} dt t^2 e^{-t^2} \quad (B1) \]
\[ N(>x) = n_e [2\pi x^{1/2} e^{-x^2} + \text{erfc} (x)] \quad (B2) \]
where $x = (T/T_e)^{1/2}$. Equating (B2) and (3b) and employing the asymptotic form of the error function complement
\[ \text{erfc} (x) \sim \frac{1}{\pi x^{1/2}} e^{-x^2} \quad (B3) \]
we obtain the approximate relation
\[ \frac{16\pi^{3/2} (T_e)^{1/2}}{3} \frac{T_e (T_e)}{P_e} 6 \times 10^7 \gg (1 + 2x^2)e^{-x^2} \quad (B4) \]
and the equality sign holds for $x = [V_e/a_{\text{min}}]$. $T_e$ is in units of eV. Equation (B4) is a general result and we illustrate the use of it for the models developed in section 4. We substitute (25) into (B4) and use the model $T_e = 2.25 \times 10^{-2} R^{1/3}$ eV to obtain
\[ R > [3.9 \times 10^3 (1 + 2x^2)e^{-x^2}]^{1/4} \quad (B5) \]
Equation (B5) has the values $R > 16, 10, 5.5$ for $x = 2, 2.5, 3$. Thus, we see that our choice of $x = 3$ throughout this paper is justified for $R > 6$. However, smaller values of $x$ occur only at greater distances. Put another way, the hotter electron temperature at greater distances requires more driving electrons relative to the background and the superthermal distribution protrudes deeper into the Maxwellian. We conclude that according to this criterion the peak that show a pronounced peak near $\omega = 1.2$ and require $x = 2.5$ (see Figure 3) occur only at distances $R > 10$; at distances closer than this $x = 3$ is the bound and the spectral peak should progressively shift toward $\omega = 1.5$ as we go to smaller distances in the magnetico- toroidal plane.

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